International Journal of Theoretical Physics, Vol. 16, No. 3 (1977), pp. 227–231

Pseudopotentials and the Inverse Scattering Problems of the Generalized Korteweg–de Vries Equation

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Received: 27 *October* 1975

Abstract

The algebraic "prolongation structure" approach of Wahlquist and Estabrook is used to determine the various forms of inverse scattering equations known for the generalized Korteweg–de Vries equation.

1. *Pseudopotentials for the Generalized Korteweg–de Vries Equation*

Much current interest revolves about the exact solutions to non-linear evolution equations determined in the last few years by means of the inverse scattering method. Recently Wahlquist and Estabrook (1975) introduced a new method for the determination of such inverse scattering problems that offers the possibility of a more systematic approach than was previously available. In this paper we wish to illustrate the new procedure by applying it to the problem of the generalized Korteweg–de Vries equation

$$u_t + 12u^2u_x + u_{xxx} = 0 \tag{1.1}$$

For details of the method which involves differential forms we refer to Wahlquist and Estabrook (1975) and Harrison and Estabrook (1971).

If we define the variables

$$z = u_x \tag{1.2}$$

$$p = z_x \tag{1.3}$$

then the equations (1.1)–(1.3) may be associated with the set of 2-forms

$$\alpha_1 = -du \wedge dx + dp \wedge dt + 12u^2 z \, dx \wedge dt \tag{1.4}$$

$$\alpha_2 = du \wedge dt = z \, dx \wedge dt \tag{1.5}$$

$$\alpha_3 = dz \wedge dt - p\, dx \wedge dt \tag{1.6}$$

Sectioning these forms (Tanaka, 1972) into a solution manifold of equations (1.1)-(1.3) annuls these forms, and consequently our system of first-order partial differential equations may be expressed by

$$\tilde{\alpha}_1 = 0, \qquad \tilde{\alpha}_2 = 0, \qquad \tilde{\alpha}_3 = 0$$

where $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$ are the sectioned forms. Complete equivalence between the partial differential set of equations and the set of forms $\alpha_1, \alpha_2, \alpha_3$ requires the set to be closed with respect to the operation of exterior differentiation; the 2-forms $d\alpha_i$ must belong to the ring of forms generated by the α_i. This is true of the forms (1.4)-(1.6).

The method of Wahlquist and Estabrook proceeds by the determination of 1-forms

$$w = dy + F(u, z, p)\, dx + G(u, z, p)\, dt \tag{1.7}$$

having exterior derivatives within the ring of forms $\alpha_1, \alpha_2, \alpha_3, w$

$$dw = \sum_{i=1}^{3} \eta^i \alpha_i + \pi \wedge w \tag{1.8}$$

Generally there will be a set of such forms $w_k (k = 1, \ldots, n)$

$$w_k = dy_k + F^k(u, z, p)\, dx + G^k(u, z, p)\, dt \tag{1.9}$$

having the property

$$dw_r = \sum_{i=1}^{3} \eta_r{}^i \alpha_i + \sum_{k=1}^{n} \pi_r{}^k \wedge w_k \tag{1.10}$$

Condition (1.10) yields the equation

$$zG_u{}^k + pG_z{}^k - 12u^2 zG^k{}_p + G^i F^k_{y_i} - F^i G^k_{y_i} = 0 \tag{1.11}$$

The last two terms of (1.11) can be expressed as the Lie derivative

$$\underset{\mathbf{G}}{\pounds}\, \mathbf{F} = [\mathbf{G}, \mathbf{F}]$$

where we introduce the notation

$$[G, F]^k = G^i F^k_{y_i} - F^i G^k_{y_i} \tag{1.12}$$

This bracket operation has the normal properties of a Lie bracket. Equation (1.11) can then be more concisely expressed as

$$z\mathbf{G}_u + p\mathbf{G}_z - 12u^2 z\mathbf{G}_p + [\mathbf{G}, \mathbf{F}] = 0 \tag{1.13}$$

From (1.13) we can determine the form of F^k and G^k to be

$$F^k = 2x_1{}^k + 2ux_2{}^k + 3u^2 x_3{}^k \tag{1.14}$$

$$\begin{aligned} G^k = {} & -2x_2{}^k(p + 4u^3) + 3x_3{}^k(z^2 - 2up - 6u^4) \\ & + 8x_4{}^k + 8x_5{}^k u + 4x_6{}^k u^2 + 4x_7{}^k z \end{aligned} \tag{1.15}$$

The equation (1.13) then gives rise to Lie bracket relations between the various x_i. These are

$$[x_1, x_3] = 0, \qquad [x_2, x_3] = 0, \qquad [x_3, x_7] = 0$$
$$[x_1, x_2] = -x_7, \qquad [x_1, x_4] = 0, \qquad [x_3, x_6] = 0, \qquad [x_2, x_6] = -2x_7 \tag{1.16}$$

together with the relations

$$[x_1, x_5] + [x_2, x_4] = 0$$
$$[x_1, x_6] + [x_3, x_4] = 0$$

We note that some brackets relations such as $[x_4, x_7]$ do not appear and others such as $[x_1, x_5]$ and $[x_2, x_4]$ are merely constrained to add to zero. In order to find a representation of this algebraic structure we adopt the procedure of trying to complete it into a Lie algebra by using the consistency requirements of Jacobi identities. In order to do this it may be necessary in general to introduce additional generators, but in this case we can determine the following Lie algebra (1.17), which contains the original algebraic structure (1.16):

$$\begin{aligned}
&[x_1, x_2] = -x_7, && [x_1, x_5] = \lambda x_7, && [x_1, x_7] = x_5\\
&[x_2, x_4] = -\lambda x_7, && [x_2, x_6] = -2x_7, && [x_2, x_7] = x_6\\
&[x_4, x_5] = -\lambda^2 x_7, && [x_4, x_7] = -\lambda x_5, && [x_5, x_7] = -\lambda x_6\\
&[x_5, x_6] = 2\lambda x_7, && [x_6, x_7] = -2x_5 &&
\end{aligned} \tag{1.17}$$

with all other Lie brackets being zero and λ an arbitrary constant.

2. *Realizations of the Algebra*

We can simplify the problem by noting that the Lie bracket relations between x_1, x_2, x_3, x_4 are satisfied if we make the identifications

$$x_3 = 0, \qquad x_1 = -\tfrac{1}{2}x_6, \qquad x_2 = -\frac{1}{\lambda}x_5, \qquad x_4 = \frac{\lambda}{2}x_6 \tag{2.1}$$

This means that we need only consider a subalgebra generated by x_5, x_6, and x_7. The relevant relations are

$$\begin{aligned}
[x_5, x_6] &= 2\lambda x_7\\
[x_5, x_7] &= -\lambda x_6\\
[x_6, x_7] &= -2x_5
\end{aligned} \tag{2.2}$$

Two particular representations are of particular relevance.

2.1. *A Two-Dimensional Linear Representation and the Tanaka (1972) Form of the Inverse Scattering Problem.* A linear representation of the commutation relations (2.2) is given by

$$\begin{aligned} x_5 &= \frac{\lambda}{\sqrt{2}}(y_1 b_2 - y_2 b_1) \\ x_6 &= \sqrt{\lambda}(y_1 b_1 - y_2 b_2) \\ x_7 &= \frac{\sqrt{\lambda}}{\sqrt{2}}(y_1 b_2 + y_2 b_1) \end{aligned} \tag{2.3}$$

where $b_i = \partial/\partial y_i$. The corresponding Pfaffian forms are

$$\begin{aligned} w_1 &= dy_1 - [\sqrt{\lambda}\, y_1 + \sqrt{2}\, u y_2]\, dx \\ &\quad + [4(u^2 + \lambda)\sqrt{2}\, y_1 + \sqrt{2}\,(p + 4u^3 + 4\lambda u - 2\sqrt{\lambda})y_2]\, dt \\ w_2 &= dy_2 + (\sqrt{2}\, u y_1 + \sqrt{\lambda}\, y_2)\, dx \end{aligned} \tag{2.4}$$

$$+ [-\sqrt{2}\,(p + 4u\lambda + 2\sqrt{\lambda}\, z)y_1 - 4(u^2 + \lambda)\sqrt{\lambda}\, y_2]\, dt \tag{2.5}$$

On a solution manifold of the prolonged ideal we will have $\widetilde{w}_1 = 0 = \widetilde{w}_2$ and this gives rise to the equations

$$y_{1x} = \sqrt{\lambda}\, y_1 + \sqrt{2}\, u y_2 \tag{2.6}$$

$$y_{1t} = -4(u^2 + \lambda)\sqrt{\lambda}\, y_1 - \sqrt{2}\,(p + 4u^3 + 4\lambda u - 2\sqrt{\lambda}\, z)y_2 \tag{2.7}$$

$$y_{2x} = -\sqrt{\lambda}\, y_2 - \sqrt{2}\, u y_1 \tag{2.8}$$

$$y_{2t} = \sqrt{2}\,(p + 4u^3 + 4\lambda u + 2\sqrt{\lambda}\, z)y_1 + 4(u^2 + \lambda)\sqrt{\lambda}\, y_2 \tag{2.9}$$

The equations (2.6) and (2.8) can be recast in the matrix form

$$\left[\begin{pmatrix} \partial/\partial x & 0 \\ 0 & -\partial/\partial x \end{pmatrix} - \begin{pmatrix} 0 & \sqrt{2}\, u \\ \sqrt{2} u & 0 \end{pmatrix}\right] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \sqrt{\lambda} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{2.10}$$

which is the form of the inverse scattering problem determined by Tanaka (1972), and Ablowitz et al. (1973).

2.2. *A Non-linear Representation and the Miura (1968) Transformation.* A non-linear realization of the algebra (2.2) is given by

$$\begin{aligned} x_5 &= i\lambda\sqrt{2} y_1 b_1 \\ x_6 &= (y_1{}^2 - \lambda) b_1 \\ x_7 &= \frac{i}{\sqrt{2}}(y_1{}^2 + \lambda) b_1 \end{aligned} \tag{2.11}$$

with the corresponding Pfaffian

$$w_1 = dy_1 - [(y_1^2 - \lambda) + 2\sqrt{2}iyu]dx + 2\sqrt{2}i[p + 4u^3 + 4\lambda u)y_1 - i\sqrt{2}(u^2 + \lambda)(y_1^2 - \lambda) + z(y_1^2 + \lambda)]dt \tag{2.12}$$

On the solution manifold of the prolonged ideal we have $\tilde{w} = 0$, which yields the equations

$$y_{1x} = (y_1^2 - \lambda) + i2\sqrt{+2}y_1 u = (y_1^2 - \lambda) + 2\sqrt{2}iyu \tag{2.13}$$

$$y_{1t} = -i2\sqrt{2}[(p + 4u^3 + 4\lambda u)y_1 - i\sqrt{2}(u^2 + \lambda)(y_1^2 - \lambda) + z(y_1^2 + \lambda)] \tag{2.14}$$

If we define y_1 by

$$Y_1 = (y_1 + \sqrt{2}iu) \tag{2.15}$$

then Y_1 satisfies the equation

$$Y_{1x} = Y_1^2 + 2\left(u^2 + \frac{i}{\sqrt{2}}u_x\right) - \lambda \tag{2.16}$$

This is a Ricatti equation, which can be linearized by the substitution

$$Y_1 = \psi_x/\psi$$

This procedure yields the standard Schrödinger equation form

$$\psi_{xx} + (2U + \lambda)\psi = 0 \tag{2.17}$$

where $U = [u^2 + (i/\sqrt{2})u_x]$.

We can identify this equation as the inverse scattering equation for the ordinary Korteweg-de Vries equation

$$U_t + U_{xxx} + 12\,UU_x = 0 \tag{2.18}$$

for it is a well-known result due to Miura that

$$U = [u^2 + (i/\sqrt{2})u_x] \tag{2.19}$$

is a solution of the Korteweg-de Vries equation whenever u is a solution of the modified Korteweg-de Vries equation.

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